Test 3 / Numerical Mathematics 1, June 21st 2022, University of Groningen

Instructions

- You have 2 hours to complete the test. When applicable, people with special facilities have 2h20 minutes in total.
- The exam is "closed book", meaning that you can only make use of the material given to you.
- The grade will be computed as the number of obtained points, plus 1.
- All answers need to be justified using mathematical arguments.

Consider the system of ODEs for the functions $x(t), y(t)$:

$$\begin{cases} x' + xy + \exp x = -3 & x(0) > 0 \\ y' + xy^3 + 3x + \log y = 0, & y(0) > 0 \end{cases}, \tag{1}$$

(a) $\boxed{1}$ Discretize (1) in time using the β-method, and denote the discrete solutions by $x_n \approx x(t_n), y_n \approx y(t_n)$. Formulate the system of (possibly non-linear) equations for (x_{n+1}, y_{n+1}) as a vector root finding problem $W(x_{n+1}, y_{n+1}) = 0$, and give the specific form for W.

Trivial, **(0.5 pt)** for each of the equations.

(b) $\boxed{3}$ Write down the Newton iteration for computing the $(k+1)$-st iterand $x_{n+1}^{(k+1)}, y_{n+1}^{(k+1)}$ from the k-th iterand $x_{n+1}^{(k)}, y_{n+1}^{(k)}$. Give specific expressions for the vectors and matrix involved, but you do not need to explicitly invert any matrix.

The general form of the Newton iterations is **(1 pt)**:

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix}^{(k+1)} = \begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix}^{(k)} - [J(x_{n+1}^{(k)}, y_{n+1}^{(k)})]^{-1} W(x_{n+1}^{(k)}, y_{n+1}^{(k)})$$

with J the Jacobian of W.
The Jacobian of the residual with respect to x_{n+1}, y_{n+1} **(2 pt)**, **(0.5 pt)** for each entry of the Jacobian:

$$J(x, y) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} + \beta h \begin{bmatrix} -y - \exp x & -x \\ -y^3 - 3 & -3xy^2 - 1/y \end{bmatrix}$$

(c) $\boxed{1}$ Which value of β would you choose for this problem? Justify your answer from the implementation point of view only. What would be the advantages and disadvantages of your choice with respect to another value of β?

$\beta = 0$ because is simpler to implement and computationally cheaper since no Newton iterations are needed **(0.5 pt)**. If $\beta > 0$, the numerical solution is implicit and therefore computationally more expensive due to non-linear iterations **(0.5 pt)**.

Consider the ODE:

$$y'(t) = f(t, y(t)) \ , \ y(0) = t_0$$

(d) $\boxed{2}$ Prove the following expression:

$$y(t_{n+1}) - u_{n+1}^* \leq h^2 \frac{\max_{t \in [t_n, t_{n+1}]} |y''(t)|}{2}.$$

with

$$u_{n+1}^* = y(t_n) + hf(t_{n+1}, y(t_{n+1})), \ h = t_{n+1} - t_n.$$

Using Taylor's theorem **(0.5 pt)**

$$y_n = y_{n+1} - y'(t_{n+1})h + \frac{y''(\xi_n)}{2}h^2, \quad \xi_n \in (t_n, t_{n+1}),$$

and therefore **(0.5 pt)**

$$-y_{n+1} = -y_n - f(t_{n+1}, y_{n+1})h + \frac{y''(\xi_n)}{2}h^2$$

and **(0.5 pt)**

$$y_{n+1} - u^*_{n+1} = -\frac{y''(\xi_n)}{2}h^2$$

and taking the max **(0.5 pt)**.

(e) [2] Consider the case $f(t, y) = \lambda y + g(t)$. Show that

$$y(t_{n+1}) - \hat{u}_{n+1} \leq \frac{\max_{t\in[t_n,t_{n+1}]} |y''(t)|}{2} \frac{h^2}{1 - h\lambda}.$$

with

$$\hat{u}_{n+1} = y(t_n) + hf(t_{n+1}, \hat{u}_{n+1}).$$

$$\begin{aligned} y_{n+1} - \hat{u}_{n+1} &= y_{n+1} - u^*_{n+1} + u^*_{n+1} - \hat{u}_{n+1} \ \textbf{(0.5 pt)} \\ &= y_{n+1} - u^*_{n+1} + h\lambda(y_{n+1} - \hat{u}_{n+1}) \ \textbf{(0.5 pt)} \\ &= \frac{y_{n+1} - u^*_{n+1}}{1 - h\lambda} \ \textbf{(0.5 pt)} \end{aligned}$$

and replacing the value from previous question concludes the proof **(0.5 pt)**.